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Chebyshev Approximation with Non–Negative Derivative

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Let X be a compact topological space and $C(X)$ the space of continuous real functions on X with norm

$$\|h\| = \sup\{|h(x)| : x \in X\}.$$

Let $\{g_1, ..., g_n\}$ be a linearly independent subset of $C(X)$ and define

$$P(c, x) = \sum_{i=1}^{n} c_i g_i(x).$$

Let ℓ be a fixed non-negative integer. The approximation problem is, given $f \in C(X)$, to find c^* which minimizes $\|r(c, \cdot)\|$, where the residual r is $r(x) = P(c, x) - f(x)$ subject to the constraint

$$P^{(\ell)}(c, \cdot) \geqslant 0. \tag{$*$}$$

$\ell = 0$ corresponds to positive approximation. $\ell = 1$ corresponds to monotone approximation. $\ell = 2$ corresponds to convex approximation. We assume that there is at least one approximation satisfying the constraint.

At little extra cost we consider a slight generalization. Let u be a given element of $C(X)$ and let the constraint be

$$P^{(\ell)}(c, \cdot) \geqslant u.$$

Choosing $u = f$ and $\ell = 0$ gives us one-sided approximation from above.

Best approximation on finite X can be determined by linear programming, as the objective function is as for Chebyshev approximation and the constraints are linear: see Rabinowitz [4]. We consider approximation on infinite X. We use a generalization of the first algorithm of Remez, described in the text of Cheney [2, p. 96]. For convenience we define

$$Q(c, x) = P^{(\ell)}(c, x) - u(x).$$

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First Algorithm of Remez

(i) Choose a finite subset X' on which $\{g_1,..., g_n\}$ is independent and set $k = 1$.

(ii) Find a best approximation $P(c^k, \cdot)$ to f (subject to the constraint) on X^k.

(iii) Find a maximum x^k of $|r(c^k, \cdot)|$ on X.

(iv) Find a minimum y^k of $Q(c^k, \cdot)$ on X.

(v) Let $X^{k+1} = X^k \cup x^k \cup y^k$.

(vi) Add 1 to k and go to (ii).

Following Cheney we define

$$\Delta^k(c) = \max |\{r(c, x)| : x \in X^k\},$$

$$\Delta(c) = \|r(c, \cdot)\|.$$

THEOREM. *$\Delta^k(c^k) \uparrow p = \inf \Delta(c)$ over c satisfying the constraint. The sequence $\{c^k\}$ is bounded and its accumulation points minimize Δ subject to the constraint.*

Proof. The arguments of Cheney show that $\{c^k\}$ is bounded. Let b be an accumulation point, say $c^{k(j)} \to b$.

ASSERTION. $Q(b, \cdot) \geqslant 0$.

Proof of assertion. Let y_0 be an accumulation point of $y^{k(j)}$, assume $y^{k(j)} \to y^0$ by taking a subsequence if necessary. As $Q(c^{k(j)}, \cdot)$ converges uniformly to $Q(b, \cdot)$ and y^k is a minimum of $Q(c^k, \cdot)$, y^0 must be a minimum of $Q(c^k, \cdot)$, y^0 must be a minimum of $Q(b, \cdot)$. Suppose $Q(b, y^0) < -\varepsilon$; then for all j sufficiently large

$$Q(c^{k(j)+1}, y^{k(j)}) < -\varepsilon$$

which contradicts choice of $c^{k(j)+1}$ and proves the assertion.

The proof of Cheney can be used to complete the proof.

A generalization of the problem is restricted derivative approximation, in which the constraint is $u \leqslant p^{(\ell)}(c, \cdot) \leqslant v$. The case $\ell = 0$ is the classical case of restricted range approximation. The algorithm is modified to also choose z_k maximizing $P^{(\ell)}(c^k, \cdot) - v$ on X and letting $X^{k+1} = X^k \cup x^k \cup y^k \cup z^k$. In the same way that we proved $Q(b, \cdot) \geqslant 0$, it is shown that $P(b, \cdot) - v \leqslant 0$, and the arguments of Cheney apply as before.

For either the algorithm or its generalization above, the convergence proof still applies if we let $X^{k+1} \supset X^k \cup x^k \cup y^k (\cup z^k)$ permitting us to add near maxima or minima to possibly speed up convergence.

It should be noted that Lewis [3] has used discretization [2, p. 84] to approximate under constraint (*) and Chalmers [1] has given a variant of the Remez exchange algorithm for approximation with linear restrictions.

Note added in proof. After this paper was written, the author found two papers [5, 6] of Watson on special cases of the problem. The paper [7] of Watson discussed the algorithm without constraints.

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